J Stat Phys (2008) 133: 813–839
DOI 10.1007/s10955-008-9639-3

Correlation Decay and Recurrence Asymptotics for Some Robust Nonuniformly Hyperbolic Maps

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Received: 27 May 2008 / Accepted: 16 October 2008 / Published online: 31 October 2008

Abstract We study a robust class of multidimensional non-uniformly hyperbolic transformations considered by Oliveira and Viana (Ergod. Theory Dyn. Syst. 28:501–533, 2008). For an open class of Hölder continuous potentials with small variation we show that the unique equilibrium state has exponential decay of correlations and that the distribution of hitting times is asymptotically exponential. Furthermore, using that the equilibrium states satisfy a weak Gibbs property we also prove log-normal fluctuations of the return times around their average.

Keywords Decay of correlations · Equilibrium states · Non-uniform hyperbolicity · Hitting and return times

1 Introduction

The ongoing interest in the recurrence properties of deterministic dynamical systems was strongly inspired by the relevance of the subject in statistical mechanics and by early contributions of Poincaré. Indeed, if the phase space of a measure preserving dynamical system is partitioned in cells of arbitrarily small diameter then not only the orbit of almost every point returns arbitrarily close to itself as the return times in these fine scales reflect the predictability of the system. More precisely, the entropy of the system is strongly related to the exponential growth rate of the return times of typical points in these finer scales.

Return time statistics attracted considerable attention in both mathematical and physics literature. First models to study the asymptotic distribution of return times are shifts of finite type endowed with a Bernoulli measure or i.i.d. random variables, where the strong independence property guarantees that any two orbits will become independent after a very short period of time, that the distribution of hitting times is asymptotically exponential and that the return time statistics coincides with the mean hitting times statistics up to some

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neglectable error. If one tries to use this strategy in a general setting, some questions arise naturally:

1. What is the distribution of the hitting times of the system when one considers sets of arbitrary small measure?
2. How do return times oscillate around their average?

In general, an answer to these questions involves a deep knowledge of the system's chaotic features (expressed in terms of mixing properties) combined with information on the measure of sets in small scales. The remarkable fact that uniformly hyperbolic dynamical systems are (semi)conjugated to subshifts of finite type, by means of Markov partitions, allowed Sinai, Ruelle and Bowen [4, 5, 23, 26] to extract many statistical properties of the system from its codification. In particular, any equilibrium state (or measure minimizing the free energy) associated to a sufficiently regular potential is indeed Bernoulli and satisfies a Gibbs property. So, in the uniformly hyperbolic setting of [8, 15, 22] it was proved exponential distribution of the hitting times and log-normal fluctuations of the return times in the Ornstein-Weiss formula for the metric entropy (see [19]). Although uniformly hyperbolic dynamics arise in physical systems (see e.g. [17]) they do not include some relevant classes of systems including the Manneville-Pomeau transformation (phenomena of intermittency), Hénon maps and billiards with convex scatterers. We note that all the previous systems present some non-uniformly hyperbolic behavior and its relevant measure satisfies some weak Gibbs property. Nevertheless, the extension of the results on decay of correlations, hitting and return time statistics beyond the scope of uniform hyperbolicity is still a challenge and gained special attention in the few past years following the recent interest and developments on the thermodynamical formalism for nonuniformly hyperbolic transformations. So, despite the effort of many authors, a general picture is still far from complete. Some of the recent contributions, where the intricate connection between hitting and return times, speed of mixing or memory loss, dimension theory and nonuniform hyperbolicity is discussed include [1, 6, 7, 9, 10, 12–14, 16, 21, 24, 25], just to mention some of the most recent advances. The majority of these results still deal with one-dimensional real or complex dynamical systems.

Here we deal with a robust class of multidimensional non-uniformly hyperbolic transformations introduced by Oliveira and Viana [18], that contain maps obtained as deformations by isotopy from expanding transformations as the ones considered in [2, Appendix]. Despite the existence of a (possibly non-generating) Markov partition many difficulties arise from the multidimensional character of the system and the absence of bounded distortion. In particular, higher dimensional intermittency phenomena due to the presence of indifferent periodic points is allowed. Oliveira and Viana developed a thermodynamical formalism to show that there is a unique equilibrium state for every Hölder potential with small variation and, moreover, that is satisfies a weak Gibbs property. Roughly, we prove that if the temperature is large enough then the equilibrium state has exponential decay of correlations, satisfies the central limit theorem and has asymptotic exponential distribution of the hitting times associated to rectangles of the refined Markov partition. Using this we obtain that the fluctuations of the return times coincide with the ones of the Shannon-McMillan-Breiman theorem, which are log-normal by the weak Gibbs condition. Indeed, let us point out that exponential return time statistics and log-normal fluctuations of return times are robust in this nonuniformly hyperbolic setting. Our approach to obtain exponential return time statistics, although similar in flavor with [21] and [12], faces distinct difficulties that arise from the non hyperbolicity of the system. While the cornerstone in [21] was the non-Markov property of partitions defining piecewise expanding maps of the interval, our main difficulties lie in the lack of bounded distortion and that the diameter of cylinders in the Markov partition may not decrease to zero. For the sake of completeness, let us mention that Arbieto and Matheus [3] proved exponential decay of correlations the equilibrium states constructed in [18] but considered different classes of potentials and observables.

A very interesting question is to obtain exponential return time statistics to balls instead of cylinders. Although this is possible to obtain for some examples in the one-dimensional setting, the study of return time statistics to balls presents itself as a major difficulty in this higher dimension setting. If such a result could be obtained we believe that our results could be extended to the more general context of [27], where [18] is generalized and no Markov partition is assumed to exist.

Overview This work is organized in the following way. In Sect. 2 we state our main results. Some notations and tools to be used in the remaining of the paper are introduced in Sect. 3. Our starting point to study the asymptotics of hitting and return times as statistical properties of the equilibrium states is to estimate the decay of correlations, that is, the velocity at which

$$C_n(\Phi, \Psi) = \left| \int \Psi(\Phi \circ T^n)\, d\mu - \int \Phi\, d\mu \int \Psi\, d\mu \right|$$

tends to zero as $n \to \infty$ for any observable Ψ and Ψ in some reasonable space of functions. In Sect. 4 we show that, for an open class of potentials, the Ruelle-Perron-Frobenius operator satisfies the spectral gap property on a space V_θ of functions with essential bounded variation that contain Hölder continuous observables and characteristic functions at elements of the dynamically generated partitions. Consequently, we obtain exponential decay of correlations and the central limit theorem. In Sect. 5 the good mixing properties are used to deduce exponential return time statistics. Finally, Sect. 6 is devoted to the proof of log-normal fluctuations for the return times.

2 Statement of the Results

2.1 Setting

Let M denote a compact Riemannian manifold. We say that a set $E \subset M$ has *finite inner diameter* if there exists $L > 0$ such that any two points in E may be joined by a curve of length less than L contained in E. Throughout, $f : M \to M$ will denote a C^1 local diffeomorphism satisfying conditions (H1) and (H2) below:

(H1) There are $p \geq 1$, $q \geq 0$, and a family $\mathcal{Q} = \{Q_1, \dots, Q_q, Q_{q+1}, \dots, Q_{q+p}\}$ of pairwise disjoint open sets whose closures have finite inner diameter and cover the whole M, such that

- Every $f|\bar{Q}_i$ is a homeomorphism onto its image
- If $f(Q_i) \cap Q_j \neq \emptyset$ then $f(Q_i) \supset Q_j$ and, hence, $f(\bar{Q}_i) \supset \bar{Q}_j$
- There is $N \geq 1$ such that $f^N(Q_i) = M$ for every i

(H2) There are positive constants $\sigma > 1$ and $L > 0$ such that

- $\|Df(x)^{-1}\| \leq \sigma^{-1}$ for every $x \in Q_{q+1} \cup \cdots \cup Q_{q+p}$
- $\|Df(x)^{-1}\| \leq L$ for every $x \in Q_1 \cup \cdots \cup Q_q$

where L is assumed to be close to be 1 in order to satisfy the relations (3).

These conditions, which roughly mean that the transformation is expanding in some (topologically) large region of M but may admit contracting behavior in the complement, are satisfied a large class of local diffeomorphisms obtained by a local bifurcation of an expanding transformation.

We will denote by $\phi : M \to \mathbb{R}$ an α-Hölder continuous potential with small oscillation, in the sense that it satisfies

(H3a) $\sup \phi - \inf \phi < \log \deg(f) - \log q$

and condition (H3b) stated at the beginning of Sect. 4.2. These conditions are clearly satisfied by an open class of potentials containing the constant ones. Note that (H1), (H2) and (H3a) are the assumptions in [18].

2.2 Equilibrium States and Conformal Measures

Given a continuous transformation $f : M \to M$ and a continuous potential $\phi : M \to \mathbb{R}$, an invariant probability measure μ is an *equilibrium state* for f with respect to ϕ if it attains the supremum

$$P_{\text{top}}(f, \phi) = \sup \left\{ h_\eta(f) + \int \phi \, d\eta : \eta \text{ is } f\text{-invariant} \right\}$$

given by the variational principle for the pressure (see e.g. [28]). The *Ruelle-Perron-Frobenius operator* $\mathcal{L}_\phi$ is the linear operator that acts in the space $C(M)$ of continuous functions by

$$\mathcal{L}_\phi g(x) = \sum_{f(y)=x} e^{\phi(y)} g(y).$$

The action of the dual operator $\mathcal{L}_\phi^*$ on the space $\mathcal{M}(M)$ of probability measures is given by $\int g \, d\mathcal{L}_\phi^* \nu = \int \mathcal{L}_\phi g \, d\nu$ for every $g \in C(M)$. We say that a measure ν is *conformal* if there exists a strictly positive function $J_\nu f$ (Jacobian of ν with respect to f) such that $\nu(f(A)) = \int J_\nu f \, d\nu$ for every measurable set A such that $f \mid A$ is injective. It is not difficult to see that any eigenmeasure ν for $\mathcal{L}_\phi^*$ associated to a positive eigenvalue λ is a conformal measure for f and that $J_\nu f = \lambda e^{-\phi}$.

A sequence of positive integers $(n_k)_{k\geq 1}$ is *non-lacunary* if it is increasing and $n_{k+1}/n_k \to 1$ when k tends to infinity. Consider the partition $\mathcal{Q}^{(n)} = \bigvee_{j=0}^{n-1} f^{-j}\mathcal{Q}$ and let $Q_n(x)$ be the element of $\mathcal{Q}^{(n)}$ that contains x.

Definition 2.1 A probability measure ν is a *non-lacunary Gibbs measure* if there is $K > 0$ so that, for ν-almost every $x \in M$ there exists some non-lacunary sequence $(n_k)_{k\geq 1}$, depending on x, such that

$$K^{-1} \leq \frac{\nu(Q_{n_k}(x))}{\exp(-Pn_k + S_{n_k}\phi(y))} \leq K$$

for every $y \in Q_{n_k}(x)$ and every $k \geq 1$.

Finally, we recall the notion of *hyperbolic time* introduced in [2]. We say that n is a *c-hyperbolic time* for $x \in M$ if

$$\prod_{j=n-k}^{n-1} \|Df(f^j(x))^{-1}\| < e^{-ck} \quad \text{for every } 1 \leq k \leq n. \tag{1}$$

Since the constant c will be fixed below, according to (2), we will refer to these simply as hyperbolic times. We say that $Q_n \in \mathcal{Q}^{(n)}$ is an *hyperbolic cylinder* if n is a hyperbolic time for *every* point in Q_n. We denote by $\mathcal{Q}_h^{(n)}$ the set of hyperbolic cylinders of order n and by H the set of points that belong to the closure of infinitely many hyperbolic cylinders. We say that a probability measure ν is *expanding* if it satisfies $\nu(H) = 1$. The next theorem summarizes the results by Oliveira, Viana [18] in this non-uniformly hyperbolic setting:

Theorem 2.2 [18] *Assume that f is a C^1 local diffeomorphism such that* (H1) *and* (H2) *hold and $\phi : M \to \mathbb{R}$ is an Hölder continuous potential that satisfies* (H3a). *Then there exists an expanding conformal measure ν such that $\mathcal{L}_\phi^* \nu = \lambda \nu$, where λ denotes the spectral radius of the operator $\mathcal{L}_\phi$ in the space $C(M)$. Moreover, there is a unique equilibrium state μ for f with respect to ϕ, it is absolutely continuous with respect to ν and it is a non-lacunary Gibbs measure.*

Throughout, μ and ν will always denote the probability measures given above.

2.3 Statement of the Main Results

First we introduce some necessary concepts. We consider a one parameter functional space V_θ, introduced in [20], using the reference partition $\mathcal{Q}$ and the conformal measure ν. Given $\theta > 0$ and $g \in L^\infty(\nu)$ define the θ-variation of g (with respect to $\mathcal{Q}$ and ν) by

$$\mathrm{var}_\theta(g) = \sum_{n \geq 1} \theta^n \sum_{Q_n \in \mathcal{Q}^{(n)}} e^{S_n\phi(Q_n)} \overline{\mathrm{osc}}(g, Q_n),$$

where $S_n\phi(Q_n) = \sup\{\sum_{j=0}^{n-1} \phi(f^j(x)) : x \in Q_n\}$ and $\overline{\mathrm{osc}}(g, Q_n)$ is the ν-essential variation of g in Q_n defined in Sect. 3.3. Let V_θ be the space of functions with essential θ-bounded variation:

$$V_\theta = \{g \in L^\infty(\nu) : \|g\|_\theta < \infty\},$$

where $\|\cdot\|_\theta = \|\cdot\|_\infty + \mathrm{var}_\theta(\cdot)$. Paccaut [20] proved that V_θ is a Banach space. We will say that μ satisfies *exponential decay of correlations* if there exist $C > 0$ and $\xi \in (0, 1)$ such that

$$C_n(\Phi, \Psi) := \left| \int \Phi(\Psi \circ f^n) d\mu - \int \Phi d\mu \int \Psi d\mu \right| \leq C\xi^n \|\Phi\|_\theta \|\Psi\|_{L^1(\nu)},$$

for every $\Psi \in L^1(\nu)$ and every $\Phi \in V_\theta$. A linear operator L in a Banach space B is *quasi-compact* if there exists an L-invariant decomposition $B = B_0 \oplus B_1$ of the Banach space such that B_0 is finite dimensional and the spectrum of $L|_{B_0}$ is a finite number of eigenvalues of absolute value $r(L)$, and $r(L|_{B_1}) < r(L)$. If $\dim(B_0) = 1$ then we say that L has a *spectral gap*. Our first main result is the following:

Theorem A *There exists a positive θ such that $\mathcal{L}_\phi$ is quasi-compact and has a spectral gap in the space V_θ. Moreover, μ has exponential decay of correlations in V_θ and the density $d\mu/d\nu$ belongs to V_θ.*

Fix θ as above. The previous result implies that the correlation functions are summable. Consequently, the *asymptotic variance* $\sigma^2(\Phi)$ defined by

$$\sigma^2(\Phi) = \|\Phi\|_{L^1(\mu)} + 2\sum_{j=1}^{\infty} \int \Phi(\Phi \circ f^j)\, d\mu$$

is well defined for every $\Phi \in V_\theta$. Standard computations involving the spectral gap property in the previous theorem (see e.g. [29]) are enough to obtain the Central Limit Theorem:

Corollary A *Assume that $\Phi \in V_\theta$ and that the asymptotic variance $\sigma^2(\Phi)$ is nonzero. Then the distribution of the random variables*

$$\frac{1}{\sigma(\Phi)\sqrt{n}} \sum_{j=0}^{n-1} \left(\Phi \circ f^j - \int \Phi\, d\mu \right)$$

converges to the normal distribution $\mathcal{N}(0,1)$ as n tends to infinity. Moreover, $\sigma(\Phi) = 0$ if and only if there exists a measurable function $\tilde{\Phi}$ such that $\Phi = \tilde{\Phi} \circ f - \tilde{\Phi}$.

Our next aim is to study the asymptotics of hitting times. Given a set Q consider the *hitting time* τ_Q defined as

$$\tau_Q(x) = \inf\{k \geq 1 : f^k(x) \in Q\}.$$

We study the deviation of the hitting times from its average given by Kac's formula: $\int \tau_Q\, d\mu = \mu(Q)^{-1}$. Indeed, we use that the potential ϕ belongs to V_θ (see Lemma 4.9) and the good mixing properties to show exponential return time statistics for the hitting time of most cylinders.

Theorem B *There are positive constants K and β, and for every $\varepsilon > 0$ there exists $N_\varepsilon \geq 1$ such that the following holds: for any $n \geq N_\varepsilon$ there exists a subset $\mathcal{Q}_\varepsilon^{(n)}$ of n-cylinders satisfying*

(1) $\mu(\cup\{Q_n : Q_n \in \mathcal{Q}_\varepsilon^{(n)}\}) \geq 1 - \varepsilon$;
(2) *For every $Q \in \mathcal{Q}_\varepsilon^{(n)}$*

$$\sup_{t \geq 0} \left| \mu\left(\tau_Q > \frac{t}{\mu(Q)} \right) - e^{-t} \right| \leq K e^{-\beta n}.$$

This theorem asserts that the distribution of the hitting times is asymptotically exponential and that the convergence is in a strong sense for the majority of the cylinders. This is very useful to study the fluctuations of the return times around the average in Ornstein-Weiss's theorem. Since μ is ergodic, if $n \geq 1$ and

$$R_n(x) = \inf\{k \geq 1 : f^k(x) \in \mathcal{Q}^{(n)}(x)\}$$

denotes the *nth return time map* then Ornstein-Weiss's theorem asserts that

$$h_\mu(f, \mathcal{Q}) = \lim_{n\to\infty} \frac{1}{n} \log R_n(x), \quad \text{for } \mu\text{-a.e. } x.$$

We shall see later on that the diameter of $Q_n(x)$ tends to zero as $n \to \infty$ at μ-almost every x, which shows that $\mathcal{Q}$ is a generating partition for μ. We study the fluctuation of the random variables $\log R_n$ around the average $nh_\mu(f)$.

Theorem C *Assume that $\sigma^2(\phi)$ is positive. Then the following convergence in distribution holds*:

$$\frac{\log R_n - nh_\mu(f)}{\sigma(\phi)\sqrt{n}} \xrightarrow[n\to+\infty]{\mathcal{D}} \mathcal{N}(0,1),$$

where $\mathcal{N}(0,1)$ denotes the standard zero mean Gaussian.

3 Preliminaries

In this section we recall some necessary concepts that will be used later on. The reader may choose to omit this section in a first reading and to come back here when necessary.

3.1 Hyperbolic Times

Here we collect some results from Sects. 3 and 4 in [18] (see also [27]), whose proofs we shall omit. Fix $\varepsilon_0 > 0$ such that $\sup\phi - \inf\phi < \log\deg(f) - \log q - \varepsilon_0$ and set $P = \log\lambda$.

Proposition 3.1 *There exists $\gamma_1 > 0$ such that the measure $\nu(Q_n) \le e^{-\gamma_1 n}$ for every cylinder $Q_n \in \mathcal{Q}^{(n)}$. There exists $\gamma \in (0,1)$ and $c_\gamma \le \log q + \varepsilon_0$ such that the cardinality of cylinders in*

$$B(n,\gamma) = \left\{Q_n \in \mathcal{Q}^{(n)} \mid \#\{0 \le j \le n-1 : f^j(Q_n) \subset \mathcal{Q}_1 \cup \cdots \cup \mathcal{Q}_q\} > \gamma n\right\}$$

is bounded from above by $\exp(c_\gamma n)$ for every large n. Moreover, the measure $\nu(B(\gamma,n))$ decreases exponentially fast as $n \to \infty$.

We are now in a position to state the precise condition on the constant $L > 0$ in (H1) that is chosen in a *different way* from [18]. Pick $c > 0$ such that

$$\sigma^{-(1-\gamma)} < e^{-2c} < 1 \quad \text{and} \quad \log q + c\alpha + \varepsilon_0 < \log\deg(f), \tag{2}$$

where $\alpha > 0$ denotes the Hölder exponent of the potential ϕ. Assume that L is sufficiently close to one such that $(\log L)^2 \le 2c^2$,

$$\sigma^{-(1-\gamma)} L^\gamma < e^{-2c} < 1 \quad \text{and} \quad \sup\phi - \inf\phi < \log\deg(f) - \log q - m\log L. \tag{3}$$

Lemma 3.2 [18, Lemma 4.4] *There exists $\tau \in (0,1)$ such that, for any $n \ge 1$ and any $x \notin B(\gamma,n)$, there exists $l > \tau n$ and integers $1 \le n_1 < \cdots < n_l$ such that x belongs to the closure of an hyperbolic cylinder $Q_{n_i} \in \mathcal{Q}_h^{n_i}$ for every $i = 1,\ldots,l$. Furthermore, $\tau \ge 2c/A$ where $A = \log L$.*

Since $0 < \tau < 1$, our choice of c in (2) guarantees that $\log q + c\tau\alpha + \varepsilon_0 < \log\deg(f)$. The following lemma asserts backward distances contraction and a Gibbs property at hyperbolic times.

Lemma 3.3 *Given $Q_n \in \mathcal{Q}_h^{(n)}$, $1 \le j$ and x, y in the closure of Q_n,*

$$d_{f^{n-j}(\bar{Q}^n)}(f^{n-j}(x), f^{n-j}(y)) \le e^{-cj} \operatorname{diam}(\mathcal{Q}).$$

Moreover, there exists $K > 0$ (independent of n) such that every $y \in \overline{Q}_n$ satisfies

$$K^{-1} \le \frac{\nu(Q_n)}{\exp(-Pn + S_n\phi(y))} \le K.$$

As an immediate consequence we obtain that the diameter of most cylinders decrease exponentially fast. More precisely,

Corollary 3.4 *The diameter of every cylinder $Q_n \notin B(n)$ satisfies*

$$\operatorname{diam}(Q_n) \le e^{-c\tau n} \operatorname{diam}(\mathcal{Q}).$$

Proof Given $Q_n \notin B(n)$, there exists a positive integer $k \ge \tau n$ such that k is a simultaneous hyperbolic time for every point in Q_n. By the mean value theorem

$$\operatorname{diam}(Q_n) \le e^{-ck} \operatorname{diam}(f^k(Q_n)) \le e^{-c\tau n} \operatorname{diam}(\mathcal{Q}),$$

which proves the corollary. □

Remark 3.5 Observe that $\#\mathcal{Q}^{(n)} \le \#\mathcal{Q} \deg(f)^{n-1}$ for every positive integer n, as an easy consequence of the Markov property and that every point has $\deg(f)$ preimages. Indeed, given $n \in \mathbb{N}$, the Markov assumption on $\mathcal{Q}$ implies that $\mathcal{Q}^{(n)} = f^{-n+1}(\mathcal{Q})$. On the other hand, given $Q \in \mathcal{Q}$, $f^{-n}(Q)$ is the union of $\deg(f)^n$ cylinders. This shows that $\#\mathcal{Q}^{(n)} \le \#\mathcal{Q} \deg(f)^n$ for every $n \ge 1$.

We say that a measure η is *exact* if every element in the tail sigma-algebra $\mathcal{B}_\infty = \bigcap_{j\ge 0} f^{-j}\mathcal{B}$ is η-trivial in the sense that it has measure zero or one.

Lemma 3.6 *μ is exact.*

Proof This is a direct consequence of [27, Lemma 6.16]. □

3.2 Weak Gibbs Property

Since μ is absolutely continuous with respect to ν and the density $d\mu/d\nu$ is bounded away from zero and infinity (see [18]), then μ satisfies the non-lacunary Gibbs property. Here we establish a criterium that relates the decay of the first hyperbolic time map with a weak Gibbs property similar to the one introduced by Yuri [30]. Let n_1 denote the first simultaneous hyperbolic time map.

Lemma 3.7 *There are almost everywhere defined function $(K_n)_{n\ge 1}$ such that*

$$K_n^{-1}(x) \le \frac{\nu(Q_n(x))}{\exp(-Pn + S_n\phi(y))} \le K_n(x)$$

for ν-almost every x and every $y \in \overline{Q}_n(x)$, and

$$\mu\Big(x \in M : K_n(x) > a(n)\Big) \le n\mu\left(x \in M : n_1(x) > \frac{\log a(n)}{P + \sup|\phi|}\right)$$

for any sequence $a(n)$. In particular, $\lim_n \frac{1}{n} \log K_n = 0$ *almost everywhere.*

Proof The proof of this lemma goes along the same ideas in [18, Lemma 3.12]. Given $n \in \mathbb{N}$ and $x \in M$ set $K_n(x) = K \exp[(P + \sup|\phi|)(n_{i+1}(x) - n_i(x))]$, where i is a positive integer such that $n_i(x) \le n < n_{i+1}(x)$ and n_i denotes the ith simultaneous hyperbolic time map. It is not hard to show that K_n verifies the Gibbs relation above. Moreover,

$$\begin{aligned}\mu\Big(x \in M : K_n(x) > a(n)\Big) &\le \mu\left(\bigcup_i \left\{x : n_1(f^{n_i(x)}(x)) > \frac{\log a(n)}{P + \sup|\phi|}\right\}\right)\\ &\le n\mu\left(x \in M : n_1(x) > \frac{\log a(n)}{P + \sup|\phi|}\right),\end{aligned}$$

where we made use of the invariance of μ and that $n_1(f^{n_i(x)}(x)) \ge n_{i+1}(x) - n_i(x)$. The last claim in the lemma is a direct consequence of the decay estimates. □

Corollary 3.8 *There exists $a \in \mathbb{N}$ such that μ-almost every x satisfies $K_n(x) < n^a$ for all but finitely many values of n.*

Proof We use the inclusion $\{n_1 > \log a(n)\} \subset B(\gamma, \log a(n))$. If $a \in \mathbb{N}$ is large enough it follows from the previous result that

$$\mu\Big(x \in M : K_n(x) > n^a\Big) \le n \exp\left(-\frac{c_\gamma a}{P + \sup|\phi|} \log n\right) \le n^{-2},$$

which is summable. Our claim follows directly from Borel-Cantelli's lemma. □

This result gives a sufficient condition to obtain sub-exponential growth of the sequence $(K_n(x))_{n\ge1}$ in the weak Gibbs property that will be of particular interest for the proof of the log-normal fluctuations of the return times in Sect. 6.

3.3 Essential Oscillation and Variation

In this section we present some basic lemmas, needed for the proof of a Lasota-Yorke inequality in Sect. 4.2. Given $g \in L^\infty(\nu)$ and a set E we define the *essential oscillation* $\overline{\mathrm{osc}}(g, E)$ of g on the set E (with respect to ν) as

$$\overline{\mathrm{osc}}(g, E) = \nu - ess \sup\{|g(x) - g(y)| : x, y \in E\}.$$

Analogously, $\overline{\sup}(g, E)$ and $\overline{\inf}(g, E)$ will denote, respectively, the essential supremum and essential infimum of g in the set E. The following is an immediate consequence of the triangular inequality.

Lemma 3.9 *For every $g, h \in L^\infty(\nu)$ and any set E it holds that*

$$\overline{\mathrm{osc}}(gh, E) \le \overline{\mathrm{osc}}(g, E)\overline{\sup}(h, E) + \overline{\sup}(g, E)\overline{\mathrm{osc}}(h, E).$$

In the next lemma we give an estimate on the oscillation of the α-Hölder continuous potential ϕ in cylinders with positive frequency of hyperbolic times.

Lemma 3.10 *There exists $C_\phi > 0$ such that*

$$\operatorname{osc}(e^{\phi}, Q_n) \le C_\phi e^{-c\tau\alpha n} \operatorname{diam}(\mathcal{Q})^{\alpha}, \quad \textit{for every } Q_n \notin B(n).$$

Proof Observe that e^{ϕ} is an α-Hölder continuous function for some positive constant C_ϕ. Therefore, it follows from Corollary 3.4 that

$$|e^{\phi(x)} - e^{\phi(y)}| \le C_\phi \operatorname{diam}(Q_n)^{\alpha} \le C_\phi e^{-c\tau\alpha n} \operatorname{diam}(\mathcal{Q})^{\alpha}$$

for every $Q_n \notin B(\gamma, n)$ and every $x, y \in Q_n$. This proves the lemma. □

The following lemma plays a key role in the proof of the Lasota-Yorke inequality.

Lemma 3.11 *Given a positive ν-measure set E and $g \in L^{\infty}(\nu)$,*

$$\overline{\sup}(g, E) \le \overline{\operatorname{osc}}(g, E) + \frac{1}{\nu(E)} \int_E |g| \, d\nu.$$

Proof Observe that $|g(x)| \le |g(x) - g(y)| + |g(y)| \le \overline{\operatorname{osc}}(g, E) + |g(y)|$ for almost every $x, y \in E$. In particular, integrating both sides of the previous inequality with respect to y it follows that $|g(x)| \le \overline{\operatorname{osc}}(g, E) + \frac{1}{\nu(E)} \int_E |g| \, d\nu$ for ν-almost every $x \in E$. The lemma is now a direct consequence of the previous relation. □

Denote by $f^k_{Q_k}$ the restriction of f^k to the cylinder Q_k and observe that it is a bijection onto its image. When no confusion is possible we will denote by Q_{n+k} the cylinder $f^{-k}_{Q_k}(Q_n)$.

Lemma 3.12 *Given any positive integers k, n and cylinders $Q_k \in \mathcal{Q}^{(k)}$ and $Q_n \in \mathcal{Q}^{(n)}$ it holds that*

$$e^{S_{n+k}\phi(f^{-k}_{Q_n}(Q_n))} \le e^{S_n\phi(Q_n)} e^{S_k\phi(f^{-k}_{Q_k}(Q_n))} \le e^{(\sup\phi - \inf\phi)k} e^{S_{n+k}\phi(f^{-k}_{Q_k}(Q_n))}.$$

Proof Fix $Q_k \in \mathcal{Q}^{(k)}$ and $Q_n \in \mathcal{Q}^{(n)}$. The first inequality is obvious. On the other hand, the Markov property implies that $f^n(Q_{n+k}) = Q_k$. If $x \in Q_n$, $y \in Q_k$ are such that attain the maximum values in $e^{S_n\phi(Q_n)}$ and $e^{S_k\phi(Q_k)}$, respectively, then

$$e^{S_{n+k}\phi(Q_{n+k})} \ge e^{S_{n+k}\phi(f^{-k}_{Q_k}(x))} = e^{S_k\phi(f^{-k}_{Q_k}(x))} e^{S_n\phi(x)}.$$

It follows immediately that

$$e^{S_k\phi(Q_k)} e^{S_n\phi(Q_n)} \le e^{S_k\phi(y) - S_k\phi(f^{-k}_{Q_k}(x))} e^{S_{n+k}\phi(Q_{n+k})} \le e^{(\sup\phi - \inf\phi)k} e^{S_{n+k}\phi(Q_{n+k})},$$

which proves the lemma. □

Since the diameter of cylinders $Q_n \notin B(n)$ decrease exponentially fast with n, the oscillation of an Hölder observable over such cylinders is also decreasing.

Lemma 3.13 *Given $k \geq 1$ there exists $C_0 > 0$ (depending on k) such that, if n is large enough then*

$$\operatorname{osc}(e^{S_k\phi}, f_{Q_k}^{-k}(Q_n)) \leq C_0 \sup(e^{S_k\phi}, f_{Q_k}^{-k}(Q_n))e^{-c\tau\alpha n}$$

for every $Q_k \in \mathcal{Q}^{(k)}$ and $Q_n \notin B(n)$.

Proof Let $k \geq 1$ and $Q_k \in \mathcal{Q}^{(k)}$ be fixed. Since ϕ is α-Hölder continuous there is $C > 0$ so that

$$|S_k\phi(x) - S_k\phi(y)| \leq \sum_{j=0}^{k-1} |\phi(f^j(x)) - \phi(f^j(y))| \leq \sum_{j=0}^{k-1} C\, d(f^j(x), f^j(y))^\alpha$$

for any $x, y \in f_{Q_k}^{-k}(Q_n)$. The term in the right hand side is clearly bounded by

$$Ck \max_{0\leq j\leq k-1} \operatorname{diam}(f^j(Q_{n+k}))^\alpha.$$

Recall that $\|Df(x)^{-1}\| \leq L$ for every $x \in M$ by (H2). In particular it follows from Corollary 3.4 that $|S_k\phi(x) - S_k\phi(y)| \leq Ck \max\{1, L^k\}^\alpha \operatorname{diam}(Q_n)^\alpha$ is arbitrarily close to zero if n is large. Since $|e^t - 1| \leq 2|t|$ for every $t \in (-1, 1)$ we conclude that $|e^{S_k\phi(x)} - e^{S_k\phi(y)}| \leq e^{S_k\phi(Q_{n+k})}|1 - e^{S_k\phi(y)-S_k\phi(x)}| \leq e^{S_k\phi(Q_{n+k})} C_0 e^{-c\tau\alpha n}$, where

$$C_0(k) = 2Ck \max\{1, L^k\}^\alpha \tag{4}$$

is independent of n. Since x and y where chosen arbitrary this completes the proof of the lemma. □

4 Spectral Gap for the Ruelle-Perron-Frobenius Operator

In this section we prove that the Ruelle-Perron-Frobenius operator has a spectral gap in the space V_θ of functions of essential bounded variation for special choices of the parameter θ. As a consequence we show that the density $d\mu/d\nu$ belongs to V_θ, and that the equilibrium state μ has exponential decay of correlations and satisfies a central limit theorem.

4.1 Continuity of the Ruelle-Perron-Frobenius Operator

For notational simplicity we denote the Ruelle-Perron-Frobenius operator $\mathcal{L}_\phi$ simply by $\mathcal{L}$. First we show that the operator $\mathcal{L}$ is continuous in the Banach space V_θ, provided that the parameter θ is small. More precisely,

Lemma 4.1 *If $\theta e^{\log\deg(f)+\sup\phi} e^{-c\tau\alpha} < 1$ then $\mathcal{L}$ is a continuous operator in V_θ: there is a positive constant C such that $\|\mathcal{L}g\|_\theta \leq C\|g\|_\theta$, for every $g \in V_\theta$.*

Proof Let $\theta > 0$ be such that $\theta e^{\log\deg(f)+\sup\phi} e^{-c\tau\alpha} < 1$. Given $g \in V_\theta$ we can write

$$\mathcal{L}g(x) = \sum_{Q\in\mathcal{Q}} e^{\phi\circ f_Q^{-1}(x)} g \circ f_Q^{-1}(x) 1_{f(Q)}(x).$$

It is clear that $\|\mathcal{L}g\|_\infty$ is bounded from above by $\#\mathcal{Q}e^{\sup\phi}\|g\|_\infty$. Thus, to prove the lemma we are reduced to show that there exists a constant $C>0$ such that

$$\mathrm{var}_\theta(\mathcal{L}g) := \sum_{n\ge 0}\theta^n \sum_{Q_n\in\mathcal{Q}^{(n)}} e^{S_n\phi(Q_n)}\overline{\mathrm{osc}}(\mathcal{L}g, Q_n) \le C\|g\|_\theta, \quad \forall g\in V_\theta. \tag{5}$$

To bound the term involving the oscillation of $\mathcal{L}g$ notice that, for every $Q_n\in\mathcal{Q}^{(n)}$

$$\begin{aligned}\overline{\mathrm{osc}}(\mathcal{L}g, Q_n) &\le \sum_{Q\in\mathcal{Q}} \overline{\mathrm{osc}}(e^{\phi\circ f_Q^{-1}} g\circ f_Q^{-1}, Q_n)\\ &\le \sum_{Q\in\mathcal{Q}} \big[\mathrm{osc}(e^\phi, f_Q^{-1}(Q_n))\overline{\sup}(g) + \sup(e^\phi)\overline{\mathrm{osc}}(g, f_Q^{-1}(Q_n))\big].\end{aligned}$$

Now we deal with the sum over elements $Q_n\in\mathcal{Q}^{(n)}$ in (5) by dividing it in two parts, according to whether Q_n belongs or not to $B(n)$. Since $\overline{\mathrm{osc}}(h)\le 2\overline{\sup}(|h|)$ for every $h\in L^\infty(\nu)$ and $\#B(\gamma,n)\le e^{(\log q+\varepsilon_0)n}$ for every large n,

$$\begin{aligned}\sum_{Q_n\in B(n)} e^{S_n\phi(Q_n)}\overline{\mathrm{osc}}(\mathcal{L}g, Q_n) &\le \#B(\gamma,n)e^{\sup(\phi)n}\times 2\#\mathcal{Q}\|e^\phi g\|_\infty\\ &\le C_1 e^{(\log q+\sup\phi+\varepsilon_0)n}\|g\|_\infty,\end{aligned}$$

for some constant C_1 depending only on ϕ and $\mathcal{Q}$. On the other hand

$$\begin{aligned}&\sum_{Q_n\notin B(n)} e^{S_n\phi(Q_n)}\overline{\mathrm{osc}(\mathcal{L}g}, Q_n)\\ &\quad\le \sum_{Q_n\notin B(n)}\sum_{Q\in\mathcal{Q}} e^{S_n\phi(Q_n)}\big[\mathrm{osc}(e^\phi, f_Q^{-1}(Q_n))\overline{\sup}(g) + \sup(e^\phi)\overline{\mathrm{osc}}(g, f_Q^{-1}(Q_n))\big].\end{aligned}$$

Lemma 3.10 implies that the right hand side above is bounded by

$$\begin{aligned}&C_2 \sum_{\substack{Q_{n+1}\in\mathcal{Q}^{(n+1)}\\ f(Q_{n+1})\notin B(n)}} e^{S_{n+1}\phi(Q_{n+1})} C_\phi e^{-c\tau\alpha n}(\mathrm{diam}\,\mathcal{Q})^\alpha\|g\|_\infty\\ &\quad + C_2 \sum_{\substack{Q_{n+1}\in\mathcal{Q}^{(n+1)}\\ f(Q_{n+1})\notin B(n)}} e^{S_{n+1}\phi(Q_{n+1})}\overline{\mathrm{osc}}(g, Q_{n+1})\end{aligned}$$

for some positive constant C_2 (depending only on ϕ). We deduce that there is $C_3>0$ (depending on ϕ, τ, α, $\mathcal{Q}$) such that $\mathrm{var}_\theta(\mathcal{L}g)$ is bounded from above by the sum of two terms:

$$\text{(a)}\quad \sum_{n=1}^\infty \theta^n\Big[C_1 e^{(\log q+\sup\phi+\varepsilon_0)n} + C_3 e^{-c\tau\alpha n}\sum_{Q_{n+1}\in\mathcal{Q}^{(n+1)}} e^{S_{n+1}\phi(Q_{n+1})}\Big]\|g\|_\infty$$

and

$$\text{(b)}\quad \frac{1}{\theta}\sum_{n=0}^\infty \theta^{n+1} C_2 \sum_{Q_{n+1}\in\mathcal{Q}^{(n+1)}} e^{S_{n+1}\phi(Q_{n+1})}\overline{\mathrm{osc}}(g, Q_{n+1}).$$

In particular it follows that $\mathrm{var}_\theta(\mathcal{L}g)$ is bounded by

$$C_2\frac{1}{\theta}\mathrm{var}_\theta(g)+\sum_{n=0}^{\infty}\Big[C_1\theta^n e^{(\log q+\sup\phi+\varepsilon_0)n}+C_3\#Q\theta^n e^{(\log\deg(f)+\sup\phi)n}e^{-c\tau\alpha n}\Big]\|g\|_\infty.$$

Our choice of the parameter θ together with the relation $\log q+c\tau\alpha+\varepsilon_0<\log\deg(f)$ guarantees the summability of the previous series and proves that $\mathcal{L}$ is a continuous operator in V_θ. □

Let us stress out that the proof of the previous lemma yields the existence of a constant $C'>0$ such that $\mathrm{var}_\theta(\mathcal{L}g)\le C_2\frac{1}{\theta}\mathrm{var}_\theta(g)+C'\|g\|_\infty$, for every small θ. However, the term $\frac{1}{\theta}\mathrm{var}_\theta(g)$ increases as θ gets smaller. In particular, the smaller θ is the higher the oscillations that may occur in elements of V_θ.

4.2 Spectral Gap and Decay of Correlations

Here we prove a Lasota-Yorke inequality for the Ruelle-Perron-Frobenius operator. This will finally imply on exponential decay of correlations and central limit theorem for the equilibrium state. Throughout, let θ be fixed and such that

$$(\star)\qquad\begin{cases}\theta e^{\log\deg(f)+\inf\phi}>L^\alpha>1,\\ \theta e^{\log\deg(f)+2\inf\phi-\sup\phi}>L^\alpha>1,\\ \theta e^{\log\deg(f)+\sup\phi}e^{-c\tau\alpha}<1.\end{cases}$$

Some comments on $(\star)$ are in order. Notice that the third condition is the one required in Lemma 4.1. On the other hand, our choice of the parameter θ impose certain restrictions on the potential ϕ. We are now in a position to state our second small variation condition on ϕ:

$$\text{(H3b)}\qquad\sup\phi-\inf\phi<\frac{1}{2}(c\tau-\log L)\alpha.$$

Remark 4.2 This condition is clearly satisfied by an open class of nearly constant potentials. Indeed, by construction, $c\tau\ge 2c^2/\log L>\log L$. Note also that (4.2) legitimates the choice in $(\star)$.

Let $r_\theta(\mathcal{L}_\phi)$ and $r(\mathcal{L}_\phi)$ denote, respectively, the spectral radius of $\mathcal{L}_\phi$ in the Banach spaces V_θ and $C(M)$. Since $\|\mathcal{L}^n 1\|_\theta\ge\|\mathcal{L}^n 1\|_\infty$ for every $n\in\mathbb{N}$ then clearly

$$r_\theta(\mathcal{L}_\phi)=\lim_{n\to\infty}(\|\mathcal{L}_\phi^n\|_\theta)^{\frac{1}{n}}\ge\lim_{n\to\infty}(\|\mathcal{L}_\phi^n 1\|_\infty)^{\frac{1}{n}}\ge\deg(f)e^{\inf\phi},$$

which proves that $\deg(f)e^{\inf\phi}$ is simultaneously a lower bound for both spectral radius $r_\theta(\mathcal{L}_\phi)$ and $r(\mathcal{L}_\phi)$. For the time being, let λ_0 denote **any** positive number larger than $\deg(f)e^{\inf\phi}$. Observe that the previous choice of θ guarantees that $\theta\lambda_0>L^\alpha>1$ and $\theta\lambda_0 e^{-(\sup\phi-\inf\phi)}>L^\alpha>1$.

Proposition 4.3 *There are B_1 and $\xi\in(0,1)$ such that, for every large $k\ge 1$ there is $B_2(k)>0$ satisfying*

$$\mathrm{var}_\theta(\lambda_0^{-k}\mathcal{L}^k g)\le B_1\xi^k\mathrm{var}_\theta(g)+B_2(k)\|g\|_{L^1(\nu)}.$$

Proof Let $k \ge 1$ be fixed and let Q_k denote the elements of the partition $\mathcal{Q}^{(k)}$. Observe that

$$\mathcal{L}^k g = \sum_{Q_k} e^{(S_k\phi)\circ f_{Q_k}^{-k}} g \circ f_{Q_k}^{-k} 1_{f^k(Q_k)}.$$

Given $g \in V_\theta$, using $\overline{\sup}(g, Q_k)$ to majorate $\overline{\sup}(g, Q_{n+k})$ and Lemma 3.11, it is not hard to see that $\mathrm{var}_\theta(\lambda_0^{-k}\mathcal{L}^k g)$ is bounded from above by the sum of the following three terms:

$$(\lambda_0\theta)^{-k}\sum_{n=0}^{\infty}\theta^{n+k}\sum_{Q_n}\sum_{Q_k} e^{S_n\phi(Q_n)}\,\mathrm{osc}(e^{S_k\phi}, Q_{n+k})\overline{\mathrm{osc}}(g, Q_k), \tag{6}$$

$$(\lambda_0\theta)^{-k}\sum_{n=0}^{\infty}\theta^{n+k}\sum_{Q_n}\sum_{Q_k} e^{S_n\phi(Q_n)}\,\mathrm{osc}(e^{S_k\phi}, Q_{n+k})\frac{1}{\nu(Q_k)}\int_{Q_k} g\,d\nu, \tag{7}$$

and

$$(\lambda_0\theta)^{-k}\sum_{n=0}^{\infty}\theta^{n+k}\sum_{Q_n}\sum_{Q_k} e^{S_n\phi(Q_n)}e^{S_k\phi(Q_{n+k})}\overline{\mathrm{osc}}(g, Q_{n+k}). \tag{8}$$

We deal with these three terms separately. First we treat (6) by rewriting it as

$$(\lambda_0\theta)^{-k}\sum_{n=0}^{\infty}\Big[\theta^n\sum_{Q_n} e^{S_n\phi(Q_n)}\Big]\Big[\theta^k\sum_{Q_k}\mathrm{osc}(e^{S_k\phi}, Q_{n+k})\overline{\mathrm{osc}}(g, Q_k)\Big]$$

and dividing the sum over elements in $\mathcal{Q}^{(n)}$ according to whether they belong or not to $B(n)$. Using $\overline{\mathrm{osc}}(g, E) \le 2\overline{\sup}(g, E)$ it follows that (6) is bounded by the sum of the two following terms:

$$(\lambda_0\theta)^{-k}\sum_{n=0}^{\infty}\Big[\theta^n\sum_{Q_n\in B(n)} e^{S_n\phi(Q_n)}\Big]\Big[\theta^k\sum_{Q_k}2\sup(e^{S_k\phi}, Q_k)\overline{\mathrm{osc}}(g, Q_k)\Big]$$

and

$$(\lambda_0\theta)^{-k}\sum_{n=0}^{\infty}\Big[\theta^n\sum_{Q_n\notin B(n)} e^{S_n\phi(Q_n)}\Big]\Big[\theta^k\sum_{Q_k}C_0(k)\sup(e^{S_k\phi}, Q_k)e^{-c\tau\alpha n}\overline{\mathrm{osc}}(g, Q_k)\Big],$$

where $C_0(k)$ is given by (4) in Lemma 3.13. Our choice on θ yields that the two previous terms are bounded from above by $C_0(k)(\lambda_0\theta)^{-k}\mathrm{var}_\theta(g)$ up to finite multiplicative constants. The constants involved are $2\sum_{n=0}^{\infty}(\theta e^{\log q+\sup\phi+\varepsilon_0})^n$ and $\sum_{n=0}^{\infty}(\theta e^{\log\deg(f)+\sup\phi}e^{-c\tau\alpha})^n$, respectively.

On the one hand, (7) is clearly bounded by $\|g\|_{L^1(\nu)}$ up to a multiplicative term obtained as the sum over all $n \ge 0$ of

$$\lambda_0^{-k}\max\nu(Q_k)^{-1}\theta^n\sum_{Q_n}\sum_{Q_k} e^{S_n\phi(Q_n)}\,\mathrm{osc}(e^{S_k\phi}, Q_{n+k}).$$

Since the measure ν gives positive weight to any cylinder in $\mathcal{Q}^{(k)}$, this shows that there exists a positive constant $K_0(k)$ such that (7) is bounded from above by $\|g\|_{L^1}$ up to the multiplicative term

$$K_0(k)\sum_{n=0}^{\infty}\theta^n\sum_{Q_n}\sum_{Q_k}e^{S_n\phi(Q_n)}\operatorname{osc}(e^{S_k\phi},Q_{n+k}).$$

The part of the sum involving elements $Q_n\in\mathcal{Q}^{(n)}$ that belong to $B(n)$ is finite because those elements grow exponentially slow compared with the allowed size of the oscillations. Indeed,

$$\sum_{n=0}^{\infty}\theta^n\sum_{Q_n\in B(n)}\sum_{Q_k}e^{S_n\phi(Q_n)}\operatorname{osc}(e^{S_k\phi},Q_{n+k})\le 2\#\mathcal{Q}^{(k)}e^{k\sup\phi}\sum_{n=0}^{\infty}\left(\theta e^{\log q+\sup\phi+\varepsilon_0}\right)^n$$

is finite. In turn, the sum over elements Q_n that do not belong $B(n)$ is also finite by Lemma 3.13:

$$\begin{aligned}
&\sum_{n=0}^{\infty}\theta^n\sum_{Q_n\notin B(n)}\sum_{Q_k}e^{S_n\phi(Q_n)}\operatorname{osc}(e^{S_k\phi},Q_{n+k})\\
&\quad\le\sum_{n=0}^{\infty}\theta^n\sum_{Q_n\notin B(n)}\sum_{Q_k}e^{S_n\phi(Q_n)}C_0e^{S_k\phi(Q_{n+k})}e^{-c\tau\alpha n}\\
&\quad\le C_0\#\mathcal{Q}^{(k)}e^{k\sup\phi}\sum_{n=0}^{\infty}\left[\theta e^{\log\deg(f)+\sup\phi}e^{-c\tau\alpha}\right]^n<\infty.
\end{aligned}$$

This shows that (7) is bounded from above by $\|g\|_{L^1}$ up to a multiplicative constant $B_2(k)$. On the other hand Lemma 3.12 ensures that (8) is bounded by

$$\begin{aligned}
&(\lambda_0\theta e^{-(\sup\phi-\inf\phi)})^{-k}\sum_{n=0}^{\infty}\theta^{n+k}\sum_{Q_{n+k}\in\mathcal{Q}^{(n+k)}}e^{S_{n+k}\phi(Q_{n+k})}\overline{\operatorname{osc}}(g,Q_{n+k})\\
&\quad\le(\lambda_0\theta e^{-(\sup\phi-\inf\phi)})^{-k}\operatorname{var}_\theta(g).
\end{aligned}$$

It follows that $\operatorname{var}_\theta(\lambda_0^{-k}\mathcal{L}^kg)\le B_1\xi^k\operatorname{var}_\theta(g)+B_2(k)\|g\|_{L^1}$, for a constant ξ is given by $\xi=\max\{(\lambda_0\theta)^{-1},(\lambda_0\theta e^{-(\sup\phi-\inf\phi)})^{-1}\}\frac{1}{k}\log C_0(k)$. Our first and second conditions on θ imply that ξ is strictly smaller than one. This completes the proof of the proposition. □

As a direct consequence we obtain the following:

Corollary 4.4 (Lasota-Yorke Inequality) *There are positive constants D_1,D_2 and $\xi_1\in(0,1)$ such that*

$$\operatorname{var}_\theta(\lambda_0^{-n}\mathcal{L}^ng)\le D_1\xi_1^n\operatorname{var}_\theta(g)+D_2\|g\|_{L^1(\nu)},$$

for every $n\ge1$.

Proof Let B_1 and ξ be given as in the previous proposition. Pick $k\ge1$ such that $\xi_1:=\sqrt[k]{B_1\xi^k}<1$ and, for any given $n\ge1$, write $n=jk+r$ where j is a positive integer and

$0 \leq r \leq k-1$. If one applies Proposition 4.3 recursively and note that $\lambda_0^{-1}\mathcal{L}$ does not increase the $L(\nu)$-norm, because ν is conformal, it follows that

$$\mathrm{var}_\theta(\lambda_0^{-n}\mathcal{L}^n g) \leq \xi_1^{kj}\,\mathrm{var}_\theta(\lambda_0^{-r}\mathcal{L}^r g) + B_2(k)\bigg(\sum_{\ell\geq 0}\xi_1^\ell\bigg)\|g\|_1.$$

Moreover, Proposition 4.3 also guarantees that

$$\mathrm{var}_\theta(\lambda_0^{-r}\mathcal{L}^n r) \leq B_1\mathrm{var}_\theta(g) + \max_{1\leq\ell\leq k} B_2(\ell)\|g\|_1.$$

The corollary is then immediate taking $D_2 = \max_{1\leq\ell\leq k} B_2(\ell) + B_2(k)(\sum_{\ell\geq 0}\xi_1^\ell)$ and $D_1 = B_1\xi_1^{-k}$. □

We proceed to prove that the Ruelle-Perron-Frobenius $\mathcal{L}_\phi$ is quasi-compact in the functional space V_θ for any parameter θ as above. Since $\lambda := r(\mathcal{L}_\phi) \geq \deg(f)e^{\inf\phi}$ the previous results hold with $\lambda_0 = \lambda$. First we show that the iterates of $\lambda^{-1}\mathcal{L}_\phi$ are well approximated by operators of finite rank. Let A_n be the linear operator defined in V_θ by

$$A_n(g) = \lambda^{-n}\mathcal{L}^n\Big(\mathbb{E}_\nu(g \mid \mathcal{Q}^{(n)})\Big)$$

for every $g \in V_\theta$, where $\mathbb{E}_\nu(\cdot \mid \mathcal{Q}^{(n)})$ stands for the conditional expectation with respect to the partition $\mathcal{Q}^{(n)}$. Since the partitions $\mathcal{Q}^{(n)}$ have finitely many elements then it is not hard to see that each A_n is a linear operator of finite rank, hence compact. In addition,

Lemma 4.5 *There is $C > 0$ and $\xi_1 \in (0,1)$ such that*

$$\|\lambda^{-n}\mathcal{L}^n - A_n\|_\theta \leq C\xi_1^n.$$

Proof First we bound the $L^\infty(\nu)$ part in $\|\cdot\|_\theta$. Given $g \in V_\theta$,

$$\begin{aligned}\|\lambda^{-n}\mathcal{L}^n g - A_n g\|_\infty &= \lambda^{-n}\|\mathcal{L}^n\big(g - \mathbb{E}(g \mid \mathcal{Q}^{(n)})\big)\|_\infty \\ &= \lambda^{-n}\bigg\|\sum_{Q_n} e^{(S_n\phi)\circ f_{Q_n}^{-n}}\big[g\circ f_{Q_n}^{-n} - \mathbb{E}_\nu(g \mid \mathcal{Q}^{(n)})\circ f_{Q_n}^{-n}\big]1_{f^n(Q_n)}\bigg\|_\infty.\end{aligned}$$

Moreover, for any $Q_n \in \mathcal{Q}^{(n)}$ the difference between g and $\mathbb{E}_\nu(g \mid \mathcal{Q}^{(n)})$ computed over the preimages of $f_{Q_n}^{-n}$ in the term above satisfies

$$\big|g\circ f_{Q_n}^{-n}(x) - \mathbb{E}_\nu(g \mid \mathcal{Q}^{(n)})\circ f_{Q_n}^{-n}(x)\big| \leq \frac{1}{\nu(Q_n)}\int_{Q_n}|g(f_{Q_n}^{-n}(x)) - g(z)|\,d\nu(z) \leq \overline{\mathrm{osc}}(g, Q_n).$$

In particular, we deduce that the L^∞ term involved in the computation of the norm $\|\cdot\|_\theta$ decreases exponentially fast:

$$\|\lambda^{-n}\mathcal{L}^n g - A_n g\|_\infty \leq \lambda^{-n}\sum_{Q_n} e^{S_n\phi(Q_n)}\overline{\mathrm{osc}}(g, Q_n) \leq (\theta\lambda)^{-n}\mathrm{var}_\theta(g).$$

The variation term in $\|\cdot\|_\theta$ can be bounded analogously, using Lemma 3.9 as in the proof of the Lasota-Yorke inequality. Indeed,

$$\begin{aligned}\mathrm{var}_\theta(\lambda^{-n}\mathcal{L}^n g - A_n g) &\le \lambda^{-n}\sum_{k=0}^{\infty}\theta^k\sum_{Q_k}\sum_{Q_n} e^{S_k\phi(Q_k)}\overline{\mathrm{osc}}(e^{S_n\phi}, Q_{n+k})\overline{\sup}(\overline{g}_n, Q_{n+k})\\ &\quad + \lambda^{-n}\sum_{k=0}^{\infty}\theta^k\sum_{Q_k}\sum_{Q_n} e^{S_k\phi(Q_k)}e^{S_n\phi(Q_{n+k})}\overline{\mathrm{osc}}(\overline{g}_n, Q_{n+k}),\end{aligned}$$

where $\overline{g}_n = g - \mathbb{E}(g \mid \mathcal{Q}^{(n)})$ and $Q_{n+k} = f_{Q_n}^{-n}(Q_k)$. Since $\mathbb{E}_\nu(g \mid \mathcal{Q}^{(n)})$ is constant over the elements in $\mathcal{Q}^{(n)}$, clearly $\overline{\mathrm{osc}}(\overline{g}_n, Q_{n+k}) = \overline{\mathrm{osc}}(g, Q_{n+k})$. In consequence the second term in the right hand side of the sum above coincides with (8), which in turn is bounded by $(\lambda\theta e^{-(\sup\phi-\inf\phi)})^{-n}\mathrm{var}_\theta(g)$. On the other hand, the first term above can be bounded as in (6) by $C(\lambda\theta L^{-\alpha})^{-n}\mathrm{var}_\theta(g)$, for some positive constant C that does not depend on n, because

$$\overline{\sup}(\overline{g}_n, Q_{n+k}) \le \overline{\sup}(\overline{g}_n, Q_n) \le \overline{\mathrm{osc}}(\overline{g}_n, Q_n) + \frac{1}{\nu(Q_n)}\int_{Q_n}\overline{g}_n\, d\nu$$

and $\int_{Q_n}\overline{g}_n\, d\nu = 0$. In consequence, there exists $C > 0$ and $0 < \xi < 1$ such that

$$\mathrm{var}_\theta(\lambda^{-n}\mathcal{L}^n g - A_n g) \le C\xi^n\|g\|_\theta.$$

Since this θ-variation term also decreases exponentially fast as n tends to infinity, this completes the proof of the lemma. □

An interesting consequence of the previous lemma is that the spectral radius of the Ruelle-Perron-Frobenius operator $\mathcal{L}_\phi$ in the Banach spaces $C(M)$ and V_θ do coincide.

Lemma 4.6 $r_\theta(\mathcal{L}_\phi) = r(\mathcal{L}_\phi)$.

Proof The spectral radius $r(\mathcal{L})$ of the linear operator $\mathcal{L}_\phi$ in the space $C(M)$ of continuous functions is clearly greater or equal than $\deg(f)e^{\inf\phi}$. Thus, the Lasota-Yorke inequality in Corollary 4.4 with $\lambda = r(\mathcal{L})$ guarantees that there exists a uniform constant $C > 0$ such that $\mathrm{var}_\theta(\lambda^{-n}\mathcal{L}^n g) \le C\|g\|_\theta$ for every $n \in \mathbb{N}$. Using $\|\cdot\|_1 \le \|\cdot\|_\infty$, this proves that there exists a uniform constant $C > 0$ such that $\|\lambda^{-n}\mathcal{L}^n g\|_\theta \le C\|g\|_\theta$ for every $g \in V_\theta$ and $n \in \mathbb{N}$. In consequence, the spectral radius $r_\theta(\lambda^{-1}\mathcal{L}_\phi)$ of $\mathcal{L}_\phi$ in V_θ verifies

$$r_\theta(\lambda^{-1}\mathcal{L}_\phi) \le 1.$$

On the other hand, using once more the conformality of the measure ν, we get

$$\|\lambda^{-n}\mathcal{L}^n 1\|_\theta \ge \|\lambda^{-n}\mathcal{L}^n 1\|_\infty \ge \|\lambda^{-n}\mathcal{L}^n 1\|_{L^1(\nu)} = 1$$

for every integer $n \ge 1$, which proves that $r_\theta(\lambda^{-1}\mathcal{L}) \ge 1$. The two estimates above imply $r_\theta(\lambda^{-1}\mathcal{L}_\phi) = \lambda^{-1}r_\theta(\mathcal{L}_\phi) = 1$, which shows that $r_\theta(\mathcal{L}_\phi) = \lambda = r(\mathcal{L}_\phi)$ and completes the proof of the lemma. □

We are now in a position to prove the quasi-compactness of the operator $\lambda^{-1}\mathcal{L}$ and, moreover, that it has a spectral gap.

Proposition 4.7 *$r_\theta(\lambda^{-1}\mathcal{L}_\phi)=1$ and the spectrum $\sigma(\lambda^{-1}\mathcal{L}_\phi)$ of the operator $\lambda^{-1}\mathcal{L}_\phi$ in V_θ satisfies*

$$\sigma(\lambda^{-1}\mathcal{L}_\phi)\subseteq\big\{z\in\mathbb{C}:|z|\le 1\big\}.$$

Moreover, 1 is a simple eigenvalue for $\lambda^{-1}\mathcal{L}_\phi$, there are no other eigenvalues of modulus one and the essential spectral radius $r_{\mathrm{ess}}(\lambda^{-1}\mathcal{L}_\phi)$ is strictly smaller than one. Furthermore, the density $d\mu/d\nu$ belongs to V_θ.

Proof Using Naussbaum's formula for the essential spectral radius (see e.g. [11, p. 709]), which asserts that

$$r_{\mathrm{ess}}(\lambda^{-1}\mathcal{L}_\phi)=\lim_{n\to\infty}(\inf\{\|\lambda^{-n}\mathcal{L}^n-L\|_\theta : L \text{ is compact operator }\})^{\frac{1}{n}},$$

and Lemma 4.5 it follows that $r_{\mathrm{ess}}(\lambda^{-1}\mathcal{L}_\phi)\le\xi_1$ is strictly smaller than one. Hence, there is only a finite number of eigenvalues with finite-dimensional eigenspaces in $\{z\in\sigma(\lambda^{-1}\mathcal{L}):|z|>r_{\mathrm{ess}}\}$. Since $r_\theta(\lambda^{-1}\mathcal{L}_\phi)=1$ there must exist some eigenvalue on the unit circle, and we can write

$$\lambda^{-1}\mathcal{L}_\phi=\Pi_1+\sum_{\substack{z\in\sigma(\lambda^{-1}\mathcal{L}_\phi)\\|z|=1}} z\Pi_z+L_0$$

where Π_z denotes the projection on the subspace associated to the eigenvalue $z\in\mathbb{C}$ and $r(L_0)<1$. Using that $\sum_{j=0}^{n-1}z^j$ is uniformly bounded in norm for every n it follows that

$$\left\|\frac{1}{n}\sum_{j=0}^{n-1}\lambda^{-j}\mathcal{L}_\phi^j-\Pi_1\right\|_\theta\xrightarrow[n\to\infty]{}0.$$

On the other hand, using $\|\cdot\|_\theta\ge\|\cdot\|_\infty\ge\|\cdot\|_{L^1(\nu)}$ one gets

$$\left\|\frac{1}{n}\sum_{j=0}^{n-1}\lambda^{-j}\mathcal{L}_\phi^j 1\right\|_\theta\ge\left\|\frac{1}{n}\sum_{j=0}^{n-1}\lambda^{-j}\mathcal{L}_\phi^j 1\right\|_{L^1(\nu)}=1,\quad\text{for every } n\in\mathbb{N}.$$

This shows that Π_1 is nonzero and that $h=\Pi_1(1)$ is an eigenfunction for $\lambda^{-1}\mathcal{L}_\phi$ associated to the eigenvalue 1. Up to a normalization in $L^1(\nu)$ it is not difficult to see that $\hat\mu=h\nu$ is an f-invariant probability measure: for every $g\in C(M)$

$$\int g\circ f\,d\hat\mu=\int\lambda^{-1}\mathcal{L}_\phi(g\circ fh)\,d\nu=\int\lambda^{-1}\mathcal{L}_\phi(h)g\,d\nu=\int g\,d\hat\mu.$$

Since $\hat\mu$ is absolutely continuous with respect to ν then it is an equilibrium state. By uniqueness of the equilibrium state, $\hat\mu$ must coincide with μ and, in particular, $d\mu/d\nu=h$ belongs to V_θ. In fact the same argument yields that 1 is a simple eigenvalue, thus, the only eigenvalue of modulus one.

It remains only to rule out the existence of other eigenvalues of modulus one distinct from 1. Let $z\in\mathbb{C}$ and $h'\in V_\theta$ be such that $\lambda^{-1}\mathcal{L}_\phi h'=zh'$ and $|z|=1$. Since h is bounded from below by some constant $C_1>0$ the function ψ defined by $h'=h\psi$ belongs to $L^2(\mu)$ because $\|\psi\|_{L^2(\mu)}\le\|h'\|_\infty^2/C_1^2<\infty$. The Koopman operator $U:L^2(\mu)\to L^2(\mu)$ acts on each $g\in L^2(\mu)$ by $U(g)=g\circ f$. By construction ψ is an eigenfunction for the dual operator U^*. Indeed, $U^*(\psi)=z\psi$ because

$$\int (U^*\psi)g\,d\mu = \int \psi(g\circ f)\,d\mu = \int h'(g\circ f)\,d\nu = \int \lambda^{-1}\mathcal{L}_\phi(h'(g\circ f))\,d\nu = \int (zh')g\,d\nu.$$

In consequence, $\psi = z^n U^n(\psi) = z^n(\psi\circ f^n)$ is measurable with respect to the sigma-algebra $f^{-n}(\mathcal{B})$ for every $n\geq 1$. Since μ is exact (recall Lemma 3.6) then ψ must be constant, which proves that h' belongs to the subspace generated by h and consequently $z=1$. This shows that 1 is the only eigenvalue of modulus one and completes the proof of the proposition. □

Corollary 4.8 *There are $C>0$, $\xi\in(0,1)$ such that every $\Phi\in L^1(\nu)$ and $\Psi\in V_\theta$ satisfy $C_n(\Phi,\Psi)\leq C\xi^n\|\Psi\|_\theta\|\Phi\|_{L^1(\nu)}$. Moreover,*

$$\left|\mu(Q_k\cap f^{-n}(Q_l)) - \mu(Q_k)\mu(Q_l)\right| \leq C\xi^n\mu(Q_l)$$

for every $n\geq 1$ and every pair of cylinders $Q_l\in\mathcal{Q}^{(l)}$ and $Q_k\in\mathcal{Q}^{(k)}$.

Proof Given $\Phi\in L^1(\nu)$ and $\Psi\in V_\theta$,

$$\int \Phi(\Psi\circ f^n)d\mu = \int \Phi h(\Psi\circ f^n)d\nu = \int (\lambda^{-n}\mathcal{L}^n)(\Phi h)\Psi\,d\nu$$

for every $n\geq 1$. Hence,

$$\left|\int \Phi(\Psi\circ f^n)d\mu - \int\Phi d\mu\int\Psi d\mu\right| = \left|\int\left[(\lambda^{-n}\mathcal{L}^n)(\Phi h) - h\int(\Phi h)\,d\nu\right]\Psi\,d\nu\right|$$
$$\leq \left\|(\lambda^{-n}\mathcal{L}^n)(\Phi h) - h\int(\Phi h)\,d\nu\right\|_\theta \|\Psi\|_{L^1(\nu)}.$$

Since $h\int(\Phi h)\,d\nu$ is the projection of Φh in the one-dimensional eigenspace associated to the eigenvalue 1 it is a consequence of the spectral gap that the previous term is bounded by $C\xi^n\|\Psi\|_\theta\|\Phi\|_{L^1(\nu)}$. On the other hand, the second claim is an immediate consequence of the first one provided that we show that the characteristic function 1_Q of a cylinder $Q\in\mathcal{Q}^{(k)}$ belongs to V_θ. Since $\overline{\text{osc}}(\cdot)\leq 2\overline{\sup}(\cdot)$, it is clear that

$$\text{var}_\theta(1_Q) = \sum_{n\geq 0}\theta^n\sum_{Q_n} e^{S_n\phi(Q_n)}\overline{\text{osc}}(1_Q,Q_n) \leq 2\sum_{n\geq 0}(\theta e^{\sup\phi})^n$$

is finite. In consequence $\|1_Q\|_\theta = 1+\text{var}_\theta(1_Q)$ is also finite, which proves our claim and finishes the proof of the corollary. □

Finally, to complete the proof of Corollary A it is enough to prove the following:

Lemma 4.9 *V_θ contains the space of α-Hölder observables.*

Proof Let g be an α-Hölder continuous observable. Since $\|g\|_\infty$ is finite it remains to estimate $\text{var}_\theta(g)$. Indeed, dividing the sum of the elements in $\mathcal{Q}^{(n)}$ according to whether they belong to $B(n)$ or not, it is not hard to check that there is $C_g>0$ such that

$$\text{var}_\theta(g) \leq 2\|g\|_\infty\sum_{n\geq 0}\theta^n\#B(n)e^{\sup\phi n} + C_g\sum_{n\geq 0}\theta^n e^{(\log\deg(f)+\sup\phi-c\tau\alpha)n}.$$

This proves the lemma. □

5 Exponential Distribution of Hitting Times

In this section we combine ideas from [12] and [21] with the weak Gibbs property and the strong mixing properties of the equilibrium state μ to study the hitting times asymptotics in Theorem B. For notational simplicity, given $Q \in \mathcal{Q}^{(n)}$ we set

$$g_Q(t) = \mu\left(\tau_Q > \frac{t}{\mu(Q)}\right).$$

The strategy to prove Theorem B is to consider a large subset $\mathcal{Q}_\varepsilon^{(n)}$ of n-cylinders such that $g_Q(\sqrt{\mu(Q)})$ behaves essentially as $\exp(-\mu(Q))$ for every $Q \in \mathcal{Q}_\varepsilon^{(n)}$ and to explore the strong mixing property of μ to obtain many instants of independence. We will need some preliminary results.

Lemma 5.1 [12, Lemma 2] *For every measurable set E and all positive t,*

$$\mu\left(\tau_E \le \frac{t}{\mu(E)}\right) \le t + \mu(E).$$

If $\gamma_1 > 0$ is given by Proposition 3.1 then we have the following:

Lemma 5.2 *There is $K > 0$ such that for any $\varepsilon > 0$ there is a subset $\mathcal{Q}_\varepsilon^{(n)}$ of n-cylinders of measure at least $1 - \varepsilon$ and satisfying*

$$e^{-\sqrt{\mu(Q)}(1+Ke^{-\gamma_1 n/2})} \le g_Q(\sqrt{\mu(Q)}) \le e^{-\sqrt{\mu(Q)}(1-Ke^{-\gamma_1 n/2})} \tag{9}$$

for every $Q \in \mathcal{Q}_\varepsilon^{(n)}$ and every large n.

Proof First observe that if n is large enough and $Q \in \mathcal{Q}^{(n)}$ arbitrary

$$\begin{aligned}-\log g_Q(\sqrt{\mu(Q)}) &= -\log\left[1 - \mu\left(\tau_Q \le \frac{\sqrt{\mu(Q)}}{\mu(Q)}\right)\right] \\ &\le \mu\left(\tau_Q \le \frac{\sqrt{\mu(Q)}}{\mu(Q)}\right) + \left[\mu\left(\tau_Q \le \frac{\sqrt{\mu(Q)}}{\mu(Q)}\right)\right]^2.\end{aligned}$$

Then Proposition 3.1 and Lemma 5.1 imply that the later sum is bounded from above by $\sqrt{\mu(Q)}(1 + Ke^{-\gamma_1 n/2})$ for some positive constant K, which proves the lower bound in (9).

In the other direction, let $\mathcal{Q}_\varepsilon^{(n)}$ be the family of n-cylinders that have no self returns in the time interval $[1, \zeta n]$ for some $\zeta > 0$: the n-cylinder Q belongs to $\mathcal{Q}_\varepsilon^{(n)}$ if $f^j(Q)$ does not intersect Q for every $1 \le j \le \zeta n$. Any element in $\mathcal{Q}^{(n)} \setminus \mathcal{Q}_\varepsilon^{(n)}$ has short recurrence and is completely characterized by ζn cylinders of the partition $\mathcal{Q}$. Consequently, $\#[\mathcal{Q}^{(n)} \setminus \mathcal{Q}_\varepsilon^{(n)}] \le (\#\mathcal{Q})^{\zeta n}$ for every n. In particular, if $\zeta = \zeta(\varepsilon)$ is chosen small enough then

$$\mu\left(\cup\{Q \in \mathcal{Q}^{(n)} : Q \notin \mathcal{Q}_\varepsilon^{(n)}\}\right) \le (\#\mathcal{Q})^{\zeta n} e^{-\gamma_1 n} < \varepsilon$$

for every large n. We claim that every $Q \in \mathcal{Q}_\varepsilon^{(n)}$ verifies the upper bound in (9). On the one hand

$$-\log g_Q(\sqrt{\mu(Q)}) \ge 1 - g_Q(\sqrt{\mu(Q)}) = \mu\left(\tau_Q \le \frac{\sqrt{\mu(Q)}}{\mu(Q)}\right).$$

Following [12, Lemma 3] and [21, Lemma 5.3], one gets

$$\mu\left(\tau_Q \le \frac{t}{\mu(Q)}\right) \ge \frac{t^2}{t^2 + \mu(Q)(1+t) + t(1+Ke^{-\gamma_1 n})}$$

for every $Q \in \mathcal{Q}_\varepsilon^{(n)}$. Together with the previous inequality this shows that

$$-\log g_Q(\sqrt{\mu(Q)}) \ge \frac{\mu(Q)}{\mu(Q)[2+\sqrt{\mu(Q)}] + \sqrt{\mu(Q)}[1+Ke^{-\gamma_1 n}]}$$

which is larger than $\sqrt{\mu(Q)}(1-Ke^{-\gamma_1 n/2})$. This completes the proof of the lemma. □

The strong mixing property of μ guarantees some independence of the system.

Lemma 5.3 *There exists $C>0$ such that if $n \ge 1$ is sufficiently large then*

$$\sup_{s \ge \sqrt{\mu(Q)}} \left| g_Q(\sqrt{\mu(Q)}+s) - g_Q(\sqrt{\mu(Q)}) g_Q(s) \right| \le C\mu(Q)^{3/4}$$

for every cylinder $Q \in \mathcal{Q}^{(n)}$.

Proof This proof is similar to the one of [21, Lemma 5.4], which explores the strong mixing properties of the system. We include a brief sketch of the proof for completeness reasons.

Let Q be any fixed cylinder of $\mathcal{Q}^{(n)}$. Given positive t, s and a small Δ (to be chosen later on), by invariance of μ it follows that $|g_Q(t+s) - g_Q(t)g_Q(s)|$ is bounded from above by the sum of the following three terms:

$$\left| g_Q(t+s) - \mu\left(\tau_Q \notin \left[0, \frac{t}{\mu(Q)}\right] \cup \left[\frac{t}{\mu(Q)} + \Delta, \frac{t+s}{\mu(Q)}\right]\right)\right|, \tag{10}$$

$$\left| \mu\left(\tau_Q \notin \left[0, \frac{t}{\mu(Q)}\right] \cup \left[\frac{t}{\mu(Q)} + \Delta, \frac{t+s}{\mu(Q)}\right]\right) - g_Q(t)\mu\left(\tau_Q \notin \left[\Delta, \frac{s}{\mu(Q)}\right]\right)\right|, \tag{11}$$

and

$$g_Q(t)\left| \mu\left(\tau_Q \notin \left[\Delta, \frac{s}{\mu(Q)}\right]\right) - \mu\left(\tau_Q \notin \left[0, \frac{s}{\mu(Q)}\right]\right)\right|. \tag{12}$$

Since (10) is the measure of the set of points that do enter Q in the time interval $[\frac{t}{\mu(Q)}, \frac{t}{\mu(Q)} + \Delta]$ then it is bounded by $\Delta\mu(Q)$. Similarly, (12) is also bounded by $\Delta\mu(Q)$. For the remaining term, computations analogous to the ones in [21, p. 356] guarantee that

$$(11) = \left| \int g_1 (g_2 \circ f^{\Delta+1})\, d\mu - \int g_1\, d\mu \int g_2\, d\mu \right| \le C\xi^{\Delta+1} \|g_1\|_\theta \|g_2\|_1,$$

where

$$g_1 = 1_{Q^c} \frac{1}{h} (\lambda^{-1}\mathcal{L}_{Q^c})^{[\frac{t}{\mu(Q)}]}(h) \quad \text{and} \quad g_2 = \prod_{j=0}^{[\frac{s}{\mu(Q)}]-\Delta-1} 1_{Q^c} \circ f^j,$$

and $\mathcal{L}_Q(g)$ stands for $\mathcal{L}(g1_Q)$. Clearly $\frac{1}{h} \in V_\theta$, because $h \in V_\theta$ and h is bounded away from zero. Observe that $\|g_2\|_1 \le 1$ and that $\|\lambda^{-1}\mathcal{L}h\|_\infty = \|h\|_\infty$ is finite. Hence, using that $\mathrm{var}_\theta(f_1 f_2) \le \mathrm{var}_\theta(f_1)\|f_2\|_\infty + \mathrm{var}_\theta(f_2)\|f_1\|_\infty$ then

$$
\begin{aligned}
(11) &\le C\xi^{\Delta+1}\left\| 1_{Q^c}\frac{1}{h}(\lambda^{-1}\mathcal{L}_{Q^c})^{[\frac{t}{\mu(Q)}]}(h)\right\|_\theta \\
&\le C\xi^{\Delta+1}\left[\mathrm{var}_\theta(\frac{1}{h})\left\|1_{Q^c}(\lambda^{-1}\mathcal{L}_{Q^c})^{[\frac{t}{\mu(Q)}]}(h)\right\|_\infty\right. \\
&\qquad \left. + \left\|\frac{1}{h}\right\|_\infty \mathrm{var}_\theta\left(1_{Q^c}(\lambda^{-1}\mathcal{L}_{Q^c})^{[\frac{t}{\mu(Q)}]}(h)\right)\right] \\
&\le C'\xi^{\Delta+1}\left[1 + \mathrm{var}_\theta\left((\lambda^{-1}\mathcal{L}_{Q^c})^{[\frac{t}{\mu(Q)}]}(h)\right)\right]
\end{aligned}
$$

for some positive constant C'. To estimate the term in the right hand side above we use (7) in [21, p. 358]: for every $N \ge 1$

$$
\mathcal{L}_{Q^c}^N(h) = h - \sum_{r=0}^{N-1}\mathcal{L}^r\mathcal{L}_Q(h) + \sum_{0\le i+j\le N-2}\mathcal{L}^i\mathcal{L}_{B_{i,j}}(h),
$$

where $B_{i,j} = Q \cap f^{-1}(Q^c) \cap \cdots \cap f^{-N+i+j+2}(Q^c) \cap f^{-N+i+j+1}(Q)$ is a cylinder of order $n + N - i - j - 1$ contained in Q. So, using the Lasota-Yorke inequality it is not hard to obtain

$$
\mathrm{var}_\theta(\lambda^{-N}\mathcal{L}_{Q^c}^N(h)) \le C''(N + N^2)\mathrm{var}_\theta(h)
$$

for some positive constant C'', and shows that

$$
|g_Q(t+s) - g_Q(t)g_Q(s)| \le 2\Delta\mu(Q) + C\xi^{\Delta+1}\left(1 + 2C''\left[\frac{t}{\mu(Q)}\right]^2\right) \le C\mu(Q)^{\frac{3}{4}}
$$

for some $C > 0$ provided that $t = \sqrt{\mu(Q)}$, $s \ge \sqrt{\mu(Q)}$ and $\Delta = \mu(Q)^{-\frac{1}{4}}$. This completes the proof of the lemma. □

We finish this section with the following:

Proof of Theorem B Let $t \ge 0$, $n \ge 1$ and $\varepsilon > 0$ be fixed, and let $\mathcal{Q}_\varepsilon^{(n)}$ be as in the previous lemma. Take $Q \in \mathcal{Q}_\varepsilon^{(n)}$ and set $k = [\frac{t}{\sqrt{\mu(Q)}}]$. The strategy is to divide the estimate on the distribution of the entrance time $g_Q(t)$ in blocks where some independence holds. Write $t = k\sqrt{\mu(Q)} + r$, with $0 \le r < \sqrt{\mu(Q)}$, and note that

$$
\begin{aligned}
(\dagger)\quad \left|g_Q(t) - e^{-t}\right| &\le \left|g_Q(t) - g_Q(k\sqrt{\mu(Q)})\right| + \left|g_Q(k\sqrt{\mu(Q)}) - g_Q(\sqrt{\mu(Q)})^k\right| \\
&\quad + \left|g_Q(\sqrt{\mu(Q)})^k - e^{-k\sqrt{\mu(Q)}}\right| + \left|e^{-k\sqrt{\mu(Q)}} - e^{-t}\right|.
\end{aligned}
$$

The last term in the right hand side above is clearly bounded from above by $2\sqrt{\mu(Q)}$, while we can use the invariance of μ to get

$$\left|g_Q(t)-g_Q(k\sqrt{\mu(Q)})\right|=\mu\left(\frac{k\sqrt{\mu(Q)}}{\mu(Q)}<\tau_Q\le\frac{t}{\mu(Q)}\right)\le\mu\left(0<\tau_Q\le\frac{\sqrt{\mu(Q)}}{\mu(Q)}\right),$$

which is bounded by $\sqrt{\mu(Q)}+\mu(Q)$ according to Lemma 5.1. Since Q belongs to $\mathcal{Q}^{(n)}_\varepsilon$ then (9) holds and the third term in (†) decays exponentially fast with n because it is bounded from above by

$$\begin{aligned}&2k\Big[-\log g_Q(\sqrt{\mu(Q)})-\sqrt{\mu(Q)}\Big]\Big[g_Q(\sqrt{\mu(Q)})^k+e^{-k\sqrt{\mu(Q)}}\Big]\\&\quad\le 2k\sqrt{\mu(Q)}Ke^{-\gamma_1 n/2}\Big[2e^{-k\sqrt{\mu(Q)}(1-Ke^{-\gamma_1 n/2})}\Big].\end{aligned}$$

Finally, we deal with the second term in the right hand side of (†). Indeed, it is not difficult (see e.g. [12, Lemma 6]) to use induction in Lemma 5.3 and obtain

$$\left|g_Q(k\sqrt{\mu(Q)})-g_Q(\sqrt{\mu(Q)})^k\right|\le C\frac{\mu(Q)^{3/4}}{1-g_Q(\sqrt{\mu(Q)})}. \tag{13}$$

Using the last inequalities in the proof of Lemma 5.2, every $Q\in\mathcal{Q}^{(n)}_\varepsilon$ satisfies $1-g_Q(\sqrt{\mu(Q)})\ge\sqrt{\mu(Q)}(1-Ke^{-\gamma_1 n})$, which guarantees that the second term in (†) satisfies

$$\left|g_Q(k\sqrt{\mu(Q)})-g_Q(\sqrt{\mu(Q)})^k\right|\le\frac{\mu(Q)^{3/4}}{\sqrt{\mu(Q)}(1-Ke^{-\gamma_1 n})}\le\mu(Q)^{1/4}$$

and decreases exponentially fast with n. This completes the proof of the theorem. □

6 Fluctuations of the Return Times

This section is devoted to the proof of Theorem C. We explore the exponential asymptotic distribution of hitting times, the weak Gibbs property of μ and the Central Limit Theorem to obtain the log-normal distribution of the return times. The following relation between hitting times and return times, similar to Lemma 4.1 in [21], is a consequence of the good mixing properties for μ.

Lemma 6.1 *Let* (t_n) *be a sequence such that* $\lim_{n\to\infty}t_n/n=+\infty$. *Then*

$$\lim_{n\to\infty}\left|\mu(R_n>t_n)-\sum_{Q\in\mathcal{Q}^{(n)}}\mu(Q)\mu(\tau_Q>t_n)\right|=0.$$

Proof Since

$$\mu(R_n>t)=\sum_{Q\in\mathcal{Q}^{(n)}}\mu(Q\cap\{\tau_Q>t\})$$

we will estimate the differences $\mu(Q\cap\{\tau_Q>t\})-\mu(Q)\mu(\tau_Q>t)$ for elements $Q\in\mathcal{Q}^{(n)}$. In fact, given $k<n<r<t$ and $Q\in\mathcal{Q}^{(n)}$ write

$$
\begin{aligned}
(*) &= \mu(Q \cap \{\tau_Q > t\}) - \mu(Q)\mu(\tau_Q > t) \\
&\le \mu(Q \cap \{\tau_Q > t\}) - \mu(Q \cap \tau_Q \notin [r,t]) \qquad (14)\\
&\quad + \mu(Q \cap \tau_Q \notin [r,t]) - \mu(Q)\mu(\tau_Q \notin [r,t]) \qquad (15)\\
&\quad + \mu(Q)\Big[\mu(\tau_Q \notin [r,t]) - \mu(\tau_Q > t)\Big]. \qquad (16)
\end{aligned}
$$

It is clear that (16) coincides with $\mu(Q)\mu(\tau_Q < r)$, which is bounded from above by $r\mu(Q)^2$, because $\mu(\tau_Q < r) \le \mu(\cup_{j\le r} f^{-j}Q)$. Moreover, by exponential decay of correlations

$$
|(15)| \le \left|\mu\left(Q \cap f^{-r}\left(\bigcap_{j=0}^{t-r} f^{-j}(Q^c)\right) - \mu(Q)\mu\left(f^{-r}\left(\bigcap_{j=0}^{t-r} f^{-j}(Q^c)\right)\right)\right)\right| \le K\xi^r.
$$

On the other hand, $|(14)|$ is bounded from above by $\sum_{j=0}^r \mu(Q \cap f^{-j}(Q))$. To deal with this last term we will consider differently whether Q belongs to the set $E_{<k}$ of n-cylinders such that there exists $0 \le i \le k$ so that $f^{-i}(Q) \cap Q \neq \emptyset$, or not. In fact, summing up over elements in $\mathcal{Q}^{(n)}$ and using the exponential decay of correlations it follows that

$$
\begin{aligned}
\sum_{Q\in\mathcal{Q}^{(n)}} |(14)| &\le \sum_{Q\in E_{<k}} \sum_{j=0}^{r} \mu(Q \cap f^{-j}(Q)) + \sum_{Q\in E_{<k}^c} \sum_{j=0}^{r} \mu(Q \cap f^{-j}(Q)) \\
&\le r\mu(E_{<k}) + \sum_{Q\in E_{<k}^c} \sum_{j=k}^{r} [K'\xi^j + \mu(Q)]\mu(Q) \\
&\le r\mu(E_{<k}) + r[K'\xi^k + e^{-\gamma_1 n}]
\end{aligned}
$$

where K' is a constant that involves the constant K from decay of correlations and an upper bound for $\|1_Q\|_\theta$. Using that $\mu(E_{<k}) \le \#\mathcal{Q}^{(k)} e^{-\gamma_1 n}$ and the previous estimates we obtain

$$
\sum_{Q\in\mathcal{Q}^{(n)}} |(*)| \le re^{-\gamma_1 n} + K\#\mathcal{Q}^{(n)}\xi^r + r\#\mathcal{Q}^{(k)} e^{-\gamma_1 n} + r[K'\xi^k + e^{-\gamma_1 n}].
$$

The previous inequality holds for $r(n) = \min\{t_n, n^2\}$ and $k(n) = [\frac{\gamma_1}{2\log\deg(f)} n]$. In fact, we get

$$
\begin{aligned}
&\left|\mu(R_n > t_n) - \sum_{Q\in\mathcal{Q}^{(n)}} \mu(Q)\mu(\tau_Q > t_n)\right| \\
&\quad \le n^2 e^{-\gamma_1 n} + K\#\mathcal{Q}\deg(f)^n \xi^{r(n)} + n^2\#\mathcal{Q} e^{-\gamma_1 n/2} + n^2[K'\xi^{k(n)} + e^{-\gamma_1 n}].
\end{aligned}
$$

The expression in the right hand side above tends to zero as $n \to \infty$. Indeed, note that the second term in the right hand side above tends to zero because $r(n)/n \to \infty$, by construction. This completes the proof of the lemma. □

Proof of Theorem C Let ϕ be an Hölder continuous potential as above such that $\sigma = \sigma(\phi) > 0$, and fix $\varepsilon > 0$ arbitrary small. Given $t \in \mathbb{R}$ and $n \ge 1$

$$\mu\left(\frac{\log R_n - nh_\mu(f)}{\sigma\sqrt{n}} > t\right) = \mu(R_n > e^{nh_\mu(f)}e^{\sigma t\sqrt{n}})$$
$$= \sum_{Q\in\mathcal{Q}^{(n)}} \mu(Q\cap\{\tau_Q > e^{nh_\mu(f)}e^{\sigma t\sqrt{n}}\}).$$

Let $\mathcal{Q}_\varepsilon^{(n)}$ be the family of cylinders given by Theorem B. Since $\cup\{Q\in\mathcal{Q}_\varepsilon^{(n)}\}$ has μ-measure at least $1-\varepsilon$ and the first entrance time τ_Q of every cylinder $Q\in\mathcal{Q}_\varepsilon^{(n)}$ has exponential distribution up to a small error, then

$$\mu\left(\frac{\log R_n - nh_\mu(f)}{\sigma\sqrt{n}} > t\right) = \sum_{Q\in\mathcal{Q}_\varepsilon^{(n)}} \mu(Q)\mu\left(\tau_Q > e^{nh_\mu(f)}e^{\sigma t\sqrt{n}}\right) + \mathcal{O}(e^{-\beta n}) + \mathcal{O}(\varepsilon).$$

This is a consequence of the previous Lemma 6.1 with $t_n = e^{nh_\mu(f)+\sigma t\sqrt{n}}$. Since $\varepsilon > 0$ was chosen arbitrary, to show the log-normal distribution of the return times we are left to prove the convergence

$$\lim_{n\to\infty}\left[\sum_{Q\in\mathcal{Q}_\varepsilon^{(n)}} \mu(Q)e^{-\mu(Q)e^{nh_\mu(f)}e^{\sigma t\sqrt{n}}}\right] = \frac{1}{\sqrt{2\pi}}\int_t^\infty e^{-\frac{x^2}{2}}\,dx + \mathcal{O}(\varepsilon). \tag{17}$$

By Lemma 3.7 and Corollary 3.8, there is $a\in\mathbb{N}$ and for μ-almost every x there is a sequence $(K_n)_n$ (depending on x) satisfying $K_n(x)\le n^a$ for all but finitely many values of n and such that $K_n(x)^{-1}e^{-Pn+S_n\phi(x)} \le \mu(Q_n(x)) \le K_n(x)e^{-Pn+S_n\phi(x)}$ for every $n\ge 1$. Using also $\mu(\cup\{Q : Q\notin\mathcal{Q}_\varepsilon^{(n)}\}) < \varepsilon$ and $h_\mu(f)+\int\phi\,d\mu = P_{\mathrm{top}}(f,\phi) = P$, for any $\rho > 0$

$$\begin{aligned}
&\sum_{Q\in\mathcal{Q}_\varepsilon^{(n)}} \mu(Q)e^{-\mu(Q)e^{nh_\mu(f)}e^{\sigma t\sqrt{n}}}\\
&= \int_{\cup\{Q:Q\in\mathcal{Q}_\varepsilon^{(n)}\}} e^{-\mu(Q(x))e^{nh_\mu(f)}e^{\sigma t\sqrt{n}}}\,d\mu(x)\\
&\ge e^{-e^{-\rho\sigma\sqrt{n}}}\left[\mu\left(x\in M \mid e^{-\mu(Q_n(x))e^{nh_\mu(f)}e^{\sigma t\sqrt{n}}} > e^{-e^{-\rho\sigma\sqrt{n}}}\right) - \varepsilon\right]\\
&\ge e^{-e^{-\rho\sigma\sqrt{n}}}\left[\mu\left(x\in M \mid \frac{-S_n\phi(x)+n\int\phi d\mu}{\sigma\sqrt{n}} > t+\rho+\frac{1}{\sqrt{n}}\log K_n(x)\right) - \varepsilon\right].
\end{aligned}$$

Since ϕ belongs to V_θ (recall Lemma 4.9) and it satisfies the Central Limit Theorem (see Corollary A), taking the limit as $n\to\infty$ and $\rho\to 0$ we obtain that

$$\liminf_{n\to\infty}\left[\sum_{Q\in\mathcal{Q}_\varepsilon^{(n)}} \mu(Q)e^{-\mu(Q)e^{nh_\mu(f)}e^{\sigma t\sqrt{n}}}\right] \ge \frac{1}{\sqrt{2\pi}}\int_t^\infty e^{-\frac{x^2}{2}}\,dx - \varepsilon.$$

The upper estimate in (17) is obtained analogously. Indeed, for any $\rho > 0$

$$\begin{aligned}
\sum_{Q\in\mathcal{Q}_\varepsilon^{(n)}} \mu(Q)e^{-\mu(Q)e^{nh_\mu(f)}e^{\sigma t\sqrt{n}}} &= \int_{\cup\{Q:Q\in\mathcal{Q}_\varepsilon^{(n)}\}} e^{-\mu(Q(x))e^{nh_\mu(f)}e^{\sigma t\sqrt{n}}}\,d\mu(x)\\
&\le \mu\left(x\in\cup\{Q : Q\in\mathcal{Q}_\varepsilon^{(n)}\} \mid e^{-\mu(Q_n(x))e^{nh_\mu(f)}e^{\sigma t\sqrt{n}}} > e^{-e^{-\rho\sigma\sqrt{n}}}\right)
\end{aligned}$$

$$\leq e^{e^{-\rho\sigma\sqrt{n}}}\left[\mu\left(x\in M\mid e^{-\mu(Q_n(x))e^{nh_\mu(f)}e^{\sigma t\sqrt{n}}}>e^{-e^{-\rho\sigma\sqrt{n}}}\right)+\varepsilon\right]$$
$$\leq e^{e^{-\rho\sigma\sqrt{n}}}\left[\mu\left(x\in M\mid \frac{-S_n\phi(x)+n\int\phi d\mu}{\sigma\sqrt{n}}>t+\rho-\frac{1}{\sqrt{n}}\log K_n(x)\right)+\varepsilon\right],$$

taking the limit as $n\to\infty$ and $\rho\to 0$ one gets

$$\limsup_{n\to\infty}\left[\sum_{Q\in\mathcal{Q}_\varepsilon^{(n)}}\mu(Q)e^{-\mu(Q)e^{nh_\mu(f)}e^{\sigma t\sqrt{n}}}\right]\leq\frac{1}{\sqrt{2\pi}}\int_t^\infty e^{-\frac{x^2}{2}}\,dx+\varepsilon,$$

which proves the upper bound in (17). The proof of the theorem is now complete. □

Acknowledgements Most of this work was done during the author's PhD thesis at IMPA, whose excellent conditions and scientific environment are greatly acknowledged. The author is grateful to V. Araújo for reading a preliminary version of this paper and making important suggestions that helped to improve the presentation of the text and to M. Todd for useful conversations. This work was partially supported by FCT-Portugal through the grant SFRH/BD/11424/2002 and by FAPERJ.

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